Numerical estimation of the Lyapunov exponents of chaotic time series corrupted by noises of large amplitude

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Abstract. We show that the Lyapunov spectrum of chaotic vector time series corrupted by noises with a noise-to-signal ratio of up to 100% in one of the coordinates can be estimated using the output of a noise reduction algorithm designed to deal with noises of large amplitude.

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INTRODUCTION

Recent research on noise reduction [1] has shown that time series of short length (up to 500 data points) generated by an smooth dynamics corrupted by observational noises of large amplitude can be detected and separated from the noise, even for noise to signal ratios of up to 300%. There are relevant characteristics of smooth dynamics, as the Lyapunov spectrum, and in particular, the negative Lyapunov exponents, whose computation is delicate even for clean time series [2, 3]. It is unclear that observed dynamics in the situation described above can be cleaned and their Lyapunov spectrum can be recovered. Such is the issue addressed in this paper.

With the purpose of estimating the Lyapunov exponents of noisy time series we first use a noise reduction algorithm that overcomes the two main intrinsic limitations of the algorithms in the literature:
(i) Early research on noise reduction [4, 5, 6, 7, 8, 9] was based on local linear estimations of the unknown dynamics using least squares methods. The inaccuracy of these methods in the local estimation of the dynamics is a known fact in the noise reduction literature [9]. We use instead the theory of *measurement error models* [10], that has recently been introduced in this area [1, 11], and gives estimators with nice statistical properties.
(ii) All noise reduction algorithms use an iterative procedure that takes the output of the algorithm as the input in the next iteration. For time series corrupted by noises of large amplitude it is crucial to have efficient criteria for determining the optimal sizes of the neighborhoods to be used in the local estimations of the dynamics at each iteration. To this end we have recently proposed [1] an adaptive neighborhoods technique based on a statistical test with null computational cost. The criterion is to keep the sizes of the neighborhoods as small as possible while guaranteeing, with a given confidence level, that the relevant information is contained in the neighborhoods. As the number of iterations increases, the noise level decreases, and the sizes of the neighborhoods reduce accordingly.

The results that we obtain with our algorithm are remarkable for short length Hénon and Lorenz time series corrupted by noises of large amplitude. Furthermore, the time series that our algorithm gives as output allows us to obtain reasonable estimates, using the classical Eckmann and Ruelle algorithm [2], of the Lyapunov exponents even for a noise having large amplitude (of up to 100% noise-to-signal ratio) in one of the components especially if the other component of the time series is corrupted by a noise of smaller amplitude. In the section devoted to numerical results we give empirical evidence of these facts for time series generated by a Henon dynamics corrupted by noise.

THE ALGORITHMS

NOISE REDUCTION ALGORITHM

We assume that the observed time series $\{\mathbf{X}_i,\ i=1,\ldots,N\}\subset\mathbb{R}^d$ is the sum of an unknown deterministic time series $\{\mathbf{s}_i, i=1,\ldots,N\}$ and an unknown $i.i.d.$ stochastic time series $\{\mathbf{e}_i, i=1,\ldots,N\}$ with null mean. Then $\mathbf{X}_i=\mathbf{s}_i+\mathbf{e}_i, 1\le i\le N,$ and $\mathbf{s}_{i+1}=f(\mathbf{s}_i)$, where $f: M\subset\mathbb{R}^d\to M$ is an unknown smooth chaotic dynamics.

Let U_i be a neighborhood of $\mathbf{X}_i$, $i=2,\ldots,N-1$, let $\{\mathbf{Z}_i:=(\mathbf{X}_{i-1},\mathbf{X}_i,\mathbf{X}_{i+1}); i=2,\ldots,N-1\}$ be a three-embedding of the noisy vector time series and $\langle\mathbf{Z}_i\rangle:=\frac{1}{\#U_i}\sum_{j:\mathbf{X}_j\in U_i}\mathbf{Z}_j$, where $\#U_i$ denotes the number of points within U_i. Since the effect of the noise is to separate the $3d$-dimensional time series $\{\mathbf{Z}_i\}$ from a d-dimensional submanifold, the noise can be partially removed by projecting the data $\{\mathbf{Z}_j - \langle\mathbf{Z}_i\rangle, \mathbf{X}_j \in U_i\}$ into optimal d-dimensional linear subspaces $\mathbf{T}_i$. The estimation of $\mathbf{z}_j := (\mathbf{s}_{j-1}, \mathbf{s}_j, \mathbf{s}_{j+1})$ obtained with U_i is $\widehat{\mathbf{z}}^i_j = \langle\mathbf{Z}_i\rangle + \mathbf{P}_{\mathbf{T}_i}(\mathbf{Z}_j - \langle\mathbf{Z}_i\rangle)$ where $\mathbf{P}_{\mathbf{T}_i}\mathbf{Z}$ denotes the orthogonal projection, with respect to the metric used to obtain $\mathbf{T}_i$, of the vector $\mathbf{Z}$ onto $\mathbf{T}_i$. The optimization criterion is the minimization of the mean square distance of the points $\mathbf{Z}_j - \langle\mathbf{Z}_i\rangle$ to the subspace taking as the metric that induced by $\widehat{\Sigma}_3^{-1}$, where $\widehat{\Sigma}_3$ is the estimation of the covariance matrix of the errors $(\mathbf{e}_{j-1}, \mathbf{e}_j, \mathbf{e}_{j+1})$ contained in $\mathbf{Z}_j$, and then,

$$\mathbf{T}_i := \arg\min_{\mathbf{T}} \sum_{j:\mathbf{X}_j \in U_i} [\mathbf{Z}_j - \langle\mathbf{Z}_i\rangle - \mathbf{P}_{\mathbf{T}}(\mathbf{Z}_j - \langle\mathbf{Z}_i\rangle)]^t\, \widehat{\Sigma}_3^{-1}\, [\mathbf{Z}_j - \langle\mathbf{Z}_i\rangle - \mathbf{P}_{\mathbf{T}}(\mathbf{Z}_j - \langle\mathbf{Z}_i\rangle)] \tag{1}$$

where $\mathbf{P}_{\mathbf{T}}\mathbf{Z} := \arg\ \min_{\mathbf{u}\in\mathbf{T}} (\mathbf{u} - \mathbf{Z})^t\ \widehat{\Sigma}_3^{-1}\ (\mathbf{u} - \mathbf{Z})$ (see in [1] the details of implementation of this algorithm). The distance induced by $\widehat{\Sigma}_3^{-1}$ takes into account that the independent variables $\mathbf{s}_j$ in the underlying linear model in the variables $(\mathbf{s}_{j-1}, \mathbf{s}_j, \mathbf{s}_{j+1})$ are also measured with error, and exploits in the optimal way the information about the structure of the error, in particular the degree of uncertainty in each of the coordinates of the time series. The solution of (1) gives unbiased and consistent estimators of the parameters of the model [10], which are also those of maximum likelihood if the errors are Gaussian.

For a clean time series, $\mathbf{T}_i$ is obtained as in (1) using the neighborhood $\{\mathbf{s}_j : (\mathbf{s}_j - \mathbf{s}_i)^t(\mathbf{s}_j - \mathbf{s}_i) \le r_0^2\}$, with r_0 guaranteeing that there are at least $d+1$ points $\mathbf{s}_j$ in the neighborhood satisfying that $\mathbf{z}_j - \langle\mathbf{z}_i\rangle$ are linearly independent. For noisy time series, the optimal subspace obtained using the neighborhood $\{\mathbf{X}_j : (\mathbf{X}_j - \mathbf{X}_i)^t(\mathbf{X}_j - \mathbf{X}_i) \le r_0^2\}$ may be far from the optimal linear subspace for the clean data. This is due to the noise, which introduces false neighbors in the neighborhoods, and it separates points which are close in the clean time series. The neighborhoods must be sufficiently large as to guarantee that a significant portion of the data within them corresponds to close neighbors for the clean time series. Furthermore, the sizes of such neighborhoods should be reduced in accordance with the noise reduction occurring as the iterative process progresses. On the other hand, since the uncertainties of the individual coordinates of the time series may be different, the Euclidean distance is not the most appropriate for the construction of the neighborhoods. We use the distance induced by $\widehat{\Sigma}_1^{-1}$ where $\widehat{\Sigma}_1$ is the estimate of the $d \times d$ covariance matrix of the error in the time series $\mathbf{X}$. Assume that $\{\mathbf{e}_i\} \sim N(0, \Sigma_1)$ and let $T^2 := (\mathbf{X}_i - \mathbf{s}_i)^t\ \widehat{\Sigma}_1^{-1}\ (\mathbf{X}_i - \mathbf{s}_i)$ where $\widehat{\Sigma}_1$ is an estimate of Σ_1. Then $\frac{(N-d)}{d(N-1)}T^2$ has an F distribution [13] with degrees of freedom d and $N-d$. A $100(1-\alpha_0)\%$ confidence ellipsoid for $\mathbf{s}_i$ is given by $\left\{\mathbf{X} : (\mathbf{X} - \mathbf{X}_i)^t\ \widehat{\Sigma}_1^{-1}\ (\mathbf{X} - \mathbf{X}_i) \le \frac{d(N-1)}{N-d} F_{d;N-d}(\alpha_0)\right\}$ where $F_{d;N-d}(\alpha_0)$ is the number such that $\Pr\{F_{d;N-d} > F_{d;N-d}(\alpha_0)\} = \alpha_0$. It is an ellipsoid centered at $\mathbf{X}_i$, whose axes are the eigenvectors of $\widehat{\Sigma}_1$ and with jth semi-axis of length $\sqrt{f_0\beta_j}$ where $\{\beta_j,\ j = 1, ..., d\}$ are the eigenvalues of $\widehat{\Sigma}_1$, and $f_0 := \frac{d(N-1)}{N-d} F_{d;N-d}(\alpha_0)$. Since we want a confidence ellipsoid for the points $\{\mathbf{s}_j : (\mathbf{s}_j - \mathbf{s}_i)^t(\mathbf{s}_j - \mathbf{s}_i) \le r_0^2\}$ we take $2\sqrt{f_0\beta_j} + r_0$ as the length of the jth semi-axis of the ellipsoid instead of $\sqrt{f_0\beta_j}$. In the examples of the next section, we use $\alpha_0 = 0.01$

LYAPUNOV EXPONENT COMPUTATION

We use the algorithm proposed by Eckmann and Ruelle [2], which is based on local linear estimates of the tangent map (see [12] for a proof of the convergence of this algorithm). It estimates the whole of the Lyapunov spectrum instead of giving only the largest Lyapunov exponent as do other algorithms [14]. Vaccari and Wang [15] have also studied the estimation of the Lyapunov spectrum of vector time series corrupted by noises of moderate amplitude.

NUMERICAL RESULTS

We show the results obtained with our algorithms for time series generated by the Hénon dynamics corrupted by an uncorrelated and heteroskedastic error with Gaussian distribution. We intend to show that the combination of the measurement error theory and the adaptive neighborhood construction allows us to reduce a noise having large amplitude in one of the components especially if the other component of the time series is corrupted by a noise of smaller amplitude. The effectiveness of the noise reduction algorithm allows us to use the output to obtain estimates of the Lyapunov spectrum.

The Hénon map is given by the equations $x_1(k+1) = 1 - 1.4x_1(k)^2 + x_2(k)$, and $x_2(k+1) = 0.3x_1(k)$. The noise level $\mathbf{NS} := ((NS)_1, (NS)_2)$ is given by the noise-to-signal ratios $(NS)_j := \sqrt{\frac{\sum_{i=1}^N \left((\mathbf{e}_i)_j\right)^2}{\sum_{i=1}^N \left((\mathbf{s}_i)_j\right)^2}}$, $j = 1, 2$ where $(\mathbf{s}_i)_j$ and $(\mathbf{e}_i)_j$ denote

TABLE 1. Noise reduction measures for noisy Hénon time series with $\mathbf{NS} = (10\%, 100\%)$.

	$N = 500$	$N = 1000$	$N = 5000$	$N = 1000$
$\langle RP \rangle$ $(\widehat{\sigma}_{R_p})$	77.62 (2.996)	81.74 (2.379)	84.77 (1.052)	85.04 (0.701)
$\langle (RP)_2 \rangle$ $(\widehat{\sigma}_{(R_p)_2})$	82.01 (3.142)	85.39 (2.348)	88.13 (0.808)	88.49 (0.676)

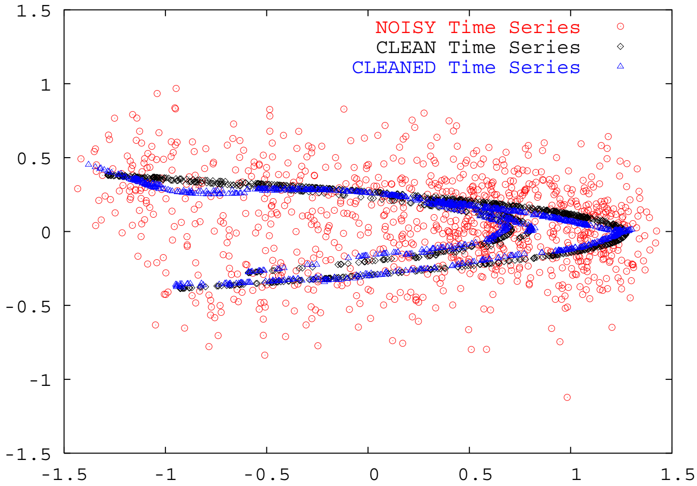

FIGURE 1. Noisy Hénon time series with $\mathbf{NS} = (10\%, 100\%)$, Clean Hénon time series and Cleaned time series.

the jth components of $\mathbf{s}_i$ and $\mathbf{e}_i$ respectively. We quantify the noise removed by the algorithm using the pointwise distance $d_P(\mathbf{s}, \widehat{\mathbf{s}}) := \left(\frac{1}{N}\sum_{i=1}^{N}(\|\widehat{\mathbf{s}}_i - \mathbf{s}_i\|_2)^2\right)^{1/2}$ between the clean $\mathbf{s}$ and the cleaned $\widehat{\mathbf{s}}$ time series. If $d_P(\mathbf{s}, \widehat{\mathbf{s}}) < d_P(\mathbf{s}, \mathbf{X})$ then the noise level in $\widehat{\mathbf{s}}$ is less than the noise level in the input time series $\mathbf{X}$. The percentage of global pointwise noise reduction is $R_p := 100\left(1 - \frac{d_P(\mathbf{s},\widehat{\mathbf{s}})}{d_P(\mathbf{s},\mathbf{X})}\right)$ and that corresponding to the jth coordinate is $(R_P)_j := 100\left(1 - \frac{d_P((\mathbf{s})_j,(\widehat{\mathbf{s}})_j)}{d_P((\mathbf{s})_j,(\mathbf{X})_j)}\right)$, $j = 1, 2$. Notice that these measures require knowledge of the clean time series. We use such knowledge only to quantify the noise level reduction. This information is used neither in the noise reduction scheme nor in deciding when the algorithm must stop. Thus the algorithm may work on data generated by an unknown process. The stopping criterion we use is the stabilization of the mean number of points in the neighborhoods.

We use a Hénon time series with a noise level $\mathbf{NS} = (10\%, 100\%)$. Thus the noise in the first component of the time series is moderate and in the second one is very large. For short length time series and large noise levels the results of the algorithm show a significant dependence on the realization of the error term, and also on the clean time series considered. For this reason we give in Table 1 the sample mean $\langle R_p \rangle$ and the sample standard deviation $\widehat{\sigma}_{R_p}$ of R_p obtained with $\max\{50000/N, 10\}$ noisy time series for $N \in \{500, 1000, 5000, 10000\}$. We quantify the level of noise reduction in the second component, which is that corrupted with a noise of larger amplitude, using $(R_p)_2$ and $\widehat{\sigma}_{(R_p)_2}$.

The results in Table 1 show values of $\langle (R_p)_2 \rangle$ higher than 82% in all the cases. The values of $\langle RP \rangle$ and $\langle (RP)_2 \rangle$ increase with N whereas $\widehat{\sigma}_{R_p}$ and $\widehat{\sigma}_{(R_p)_2}$ decreases with N. Figure 1 show the noisy, the clean, and the time series that the algorithm gives as output for one of the 50 noisy time series corresponding to $N = 1000$. The figure shows that even for a time series having such a large noise level and such a short length the algorithm is able to recover a significant part of the geometric structure of the clean time series.

The true values of the Lyapunov exponents (that can be obtained using a clean Hénon time series and the true tangent maps) are $\lambda_1 \approx 0.42$, $\lambda_2 \approx -1.62$ and $\lambda_1 + \lambda_2 = \ln(0.3) \approx -1.204$. The mean values $\langle \widehat{\lambda}_1 \rangle$ and $\langle \widehat{\lambda}_2 \rangle$ of the estimates of the Lyapunov exponents obtained using ten outputs of the noise reduction algorithm are shown in Table 2. These results show, even for short length time series, a positive Lyapunov exponent and therefore a chaotic

TABLE 2. Mean Lyapunov exponents obtained using ten outputs of the noise reduction algorithm for noisy Hénon time series with $\mathbf{NS} = (10\%, 100\%)$.

	$N=500$	$N=1000$	$N=5000$	$N=1000$
$\left\langle \widehat{\lambda}_1 \right\rangle$ $(\widehat{\sigma}_{\widehat{\lambda}_1})$	0.503 (0.048)	0.443 (0.030)	0.410 (0.014)	0.403 (0.013)
$\left\langle \widehat{\lambda}_2 \right\rangle$ $(\widehat{\sigma}_{\widehat{\lambda}_2})$	-1.105 (0.239)	-1.334 (0.198)	-1.528 (0.068)	-1.559 (0.089)

dynamics, and the contraction of volumes elements reflected by a negative sum of the Lyapunov exponents. The results are reasonable estimates of both Lyapunov exponents for large length time series, and improve those obained in [15] for a Lorenz dynamics for a moderate noise level $\mathbf{NS} = (13\%, 7.2\%, 8.1\%)$ using a multivariate global polynomial regression model. We have also designed a test in order to determine whether the existence of a positive Lyapunov exponent is an evidence of chaotic dynamics or not. The interpretation of the Lyapunov exponents as asymptotic exponential rates of convergence or divergence of trajectories corresponding to nearby initial conditions is based on the existence of a differentiable dynamics. In this case the mean square error SRR_i made by the algorithm at the point $\mathbf{X}_i$ using as neighborhood V_i the NV closest points to $\mathbf{X}_i$ is small if NV is small and N is large. Then the quantity $R_i := 1 - \frac{SRR_i}{SYY_i}$ where $SYY_i := \sum_{j:\mathbf{X}_j \in V_i} \left\|\mathbf{X}_{j+1}\right\|^2$ satisfies $0 \leq R_i \leq 1$, and it is close to one when the linear estimation is good. We consider as measure of goodness of the linear fits $R := \frac{1}{N-1}\sum_{i=1}^{N-1} R_i$. In all the results we present above the mean value of R is up to 0.95. However, if we calculate the Lyapunov exponents taking as input just the Gaussian noise used to corrupt the Hénon time series, we observe that the positive Lyapunov exponent with has a similar vale to the positive Lyapunov exponent of the Hénon dynamics. However the estimates depend strongly on N, NV and the values of R are very small. For instance for a time series with $N = 5000$ and $NV = 100$, we obtain $\widehat{\lambda}_1 = 0.524$, $\widehat{\lambda}_2 = -0.699$ and $R = 0.09$, and for $NV = 200$ the estimates are $\widehat{\lambda}_1 = 0.471, \widehat{\lambda}_2 = -0.95$ and $R = 0.14$. Analogous results are obtained if we estimate the Lyapunov exponents using the noisy Hénon time series for the same lenght $N = 5000$: $\widehat{\lambda}_1 = 0.870$, $\widehat{\lambda}_2 = -0.258$ and $R = 0.26$ using $NV = 100$, and $\widehat{\lambda}_1 = 1.693$, $\widehat{\lambda}_2 = 0.551$ and $R = 0.45$ with $NV = 200$. In these two cases we reject the positive Lyapunov exponent as an evidence of chaotic dynamics.

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